Biot Theory (Almost) For Dummies

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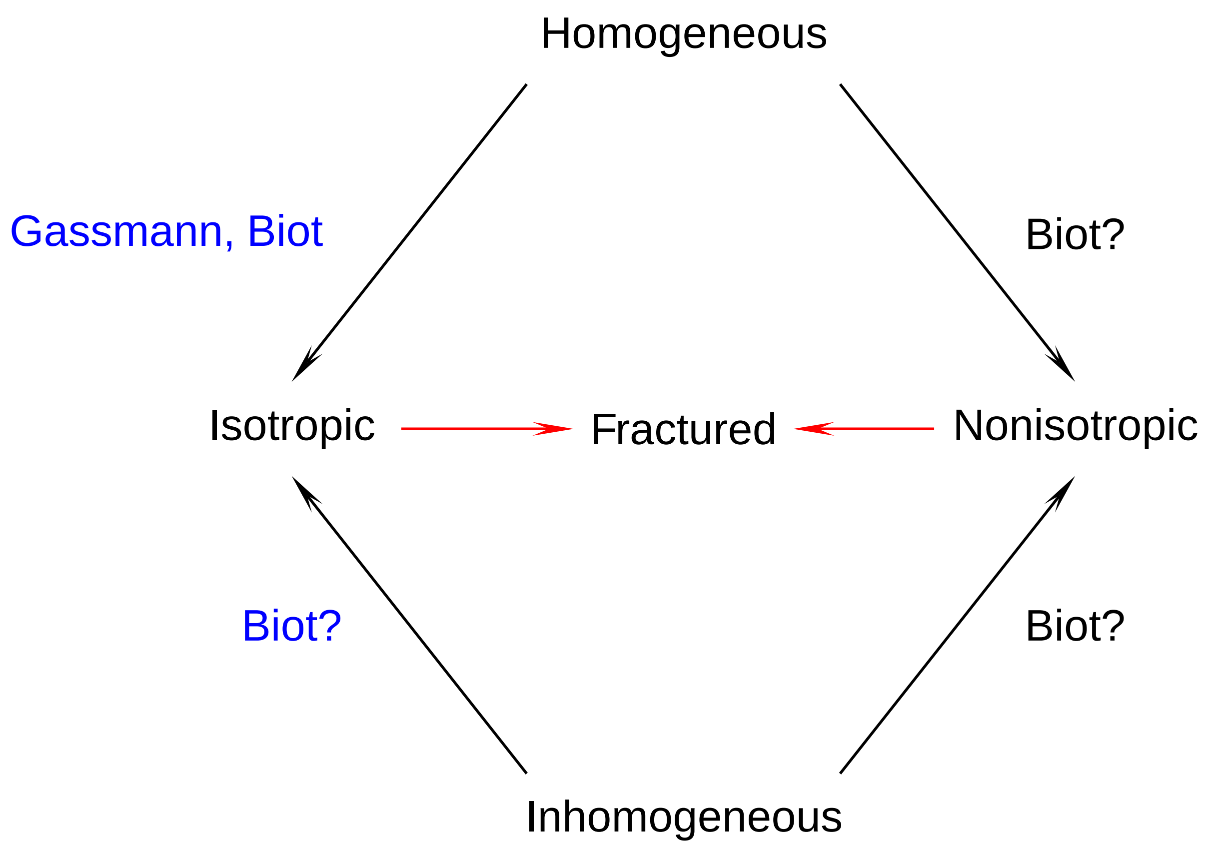
December 5, 2005, Seminar at the University of Houston

Rock Classification



Rock Types

Homogeneous, isotropic

Heterogeneous, isotropic

GASSMANN's theory works only for the microscopically homogeneous rock (e.g., uniform spheres)

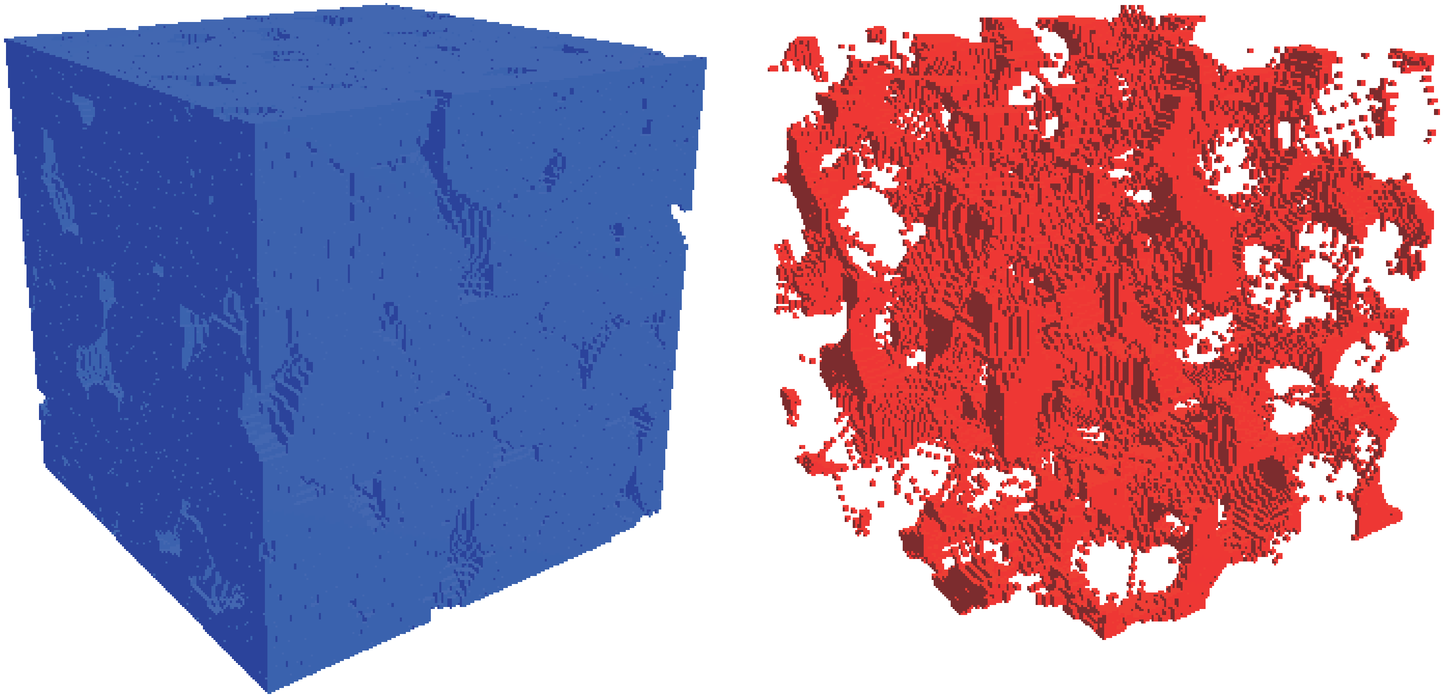
Rock Types

It is impossible to use equivalent homogeneous rock to explain heterogeneous rocks. This is especially true for clay-rich rocks, ZOBACK & BEYERLEE (1975), BERRYMAN, (1992)

(In)homogeneous, anisotropic

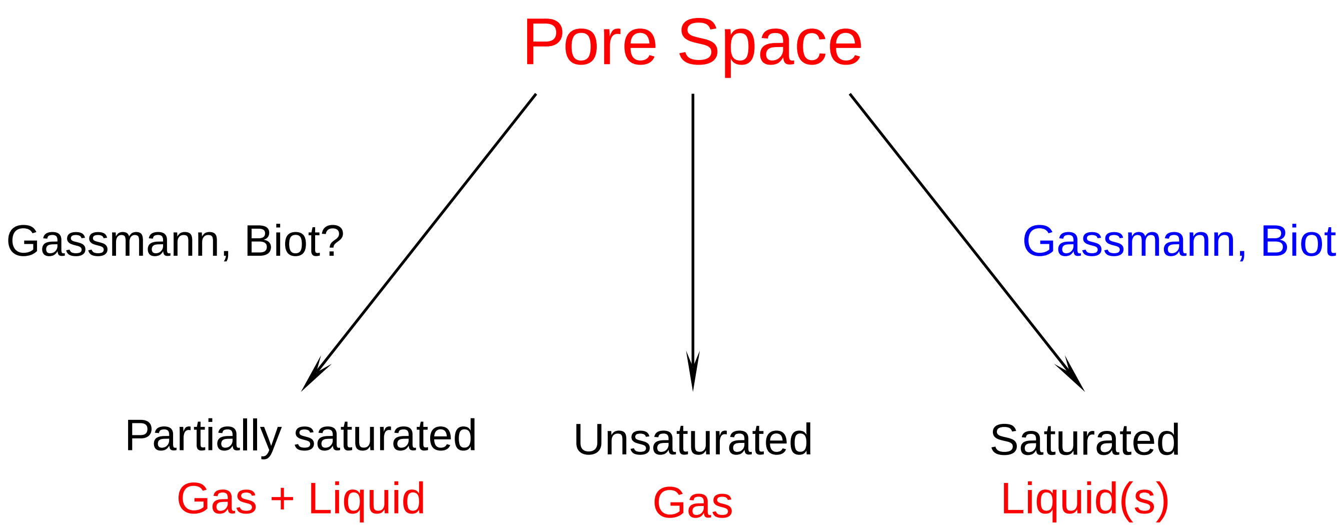
A new theory must be developed for fractured, heterogeneous rocks

Porous Rock



Porous rock = **Solid Skeleton** + **Pore Space**

Porous Rock Characterization



Bulk density

$$\rho = \frac{\text{mass of solid skeleton} + \text{mass of pore space fluids}}{\text{bulk volume of rock}}$$

$$\rho = (1-\phi)\rho_s + \phi\rho_f = \rho_{\text{skeleton}} + \phi\rho_f$$

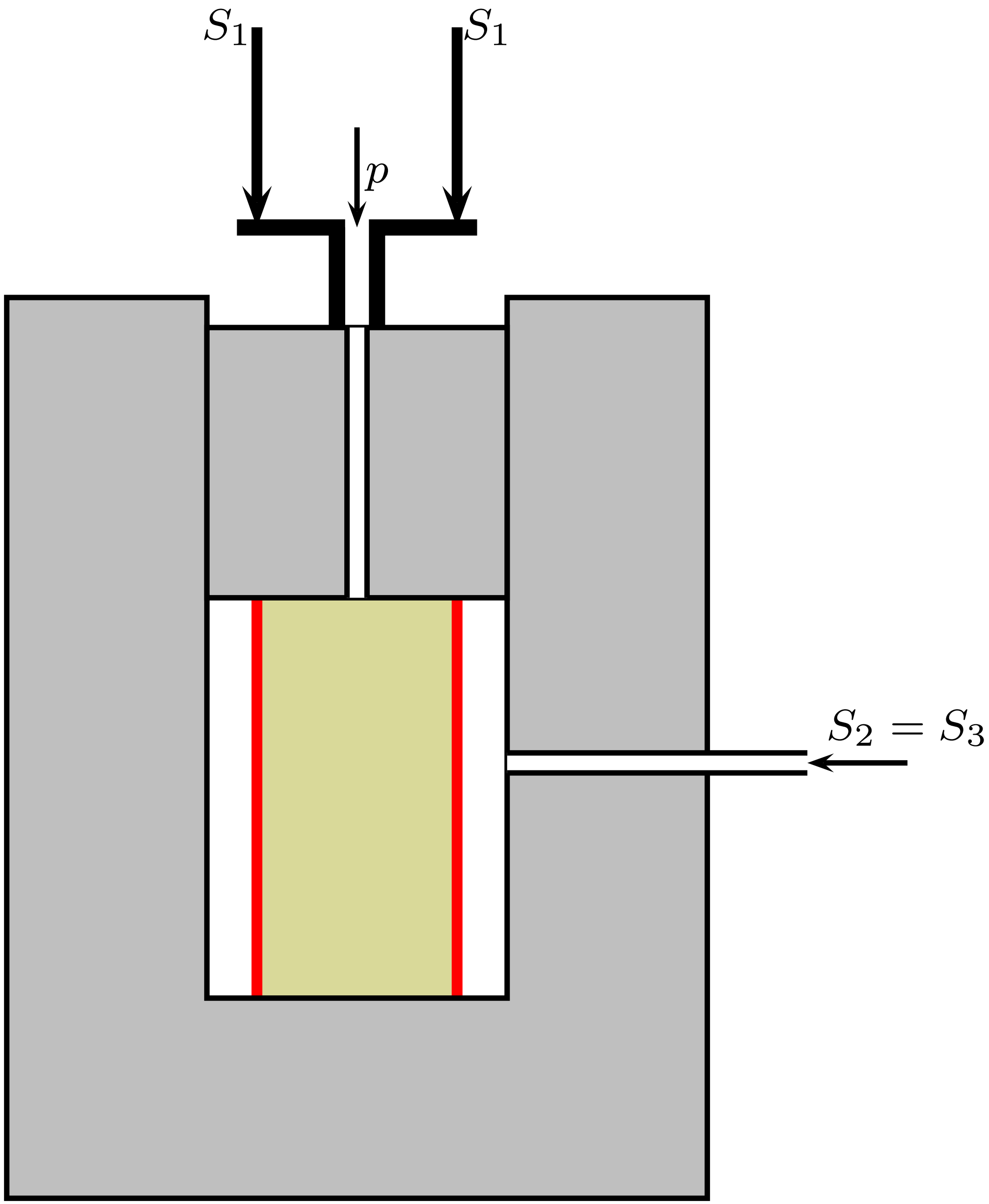
Compressibility Measurements

The vertical stress, S_1, is applied to a hollow piston. The tube in the piston is used to regulate the pore pressure, p. The lateral stresses, $S_2 = S_3$, are applied to the copper-jacketed specimen by injecting oil through the side tube. The confining pressure is defined as

$$p_c = -\sigma = \frac{1}{3}(S_1 + S_2 + S_3)$$

The jacketed or drained triaxial rock compressibility:

$$\beta := -\frac{1}{V}\left(\frac{\partial V}{\partial p_c}\right)_{p,T} = \frac{1}{K}$$



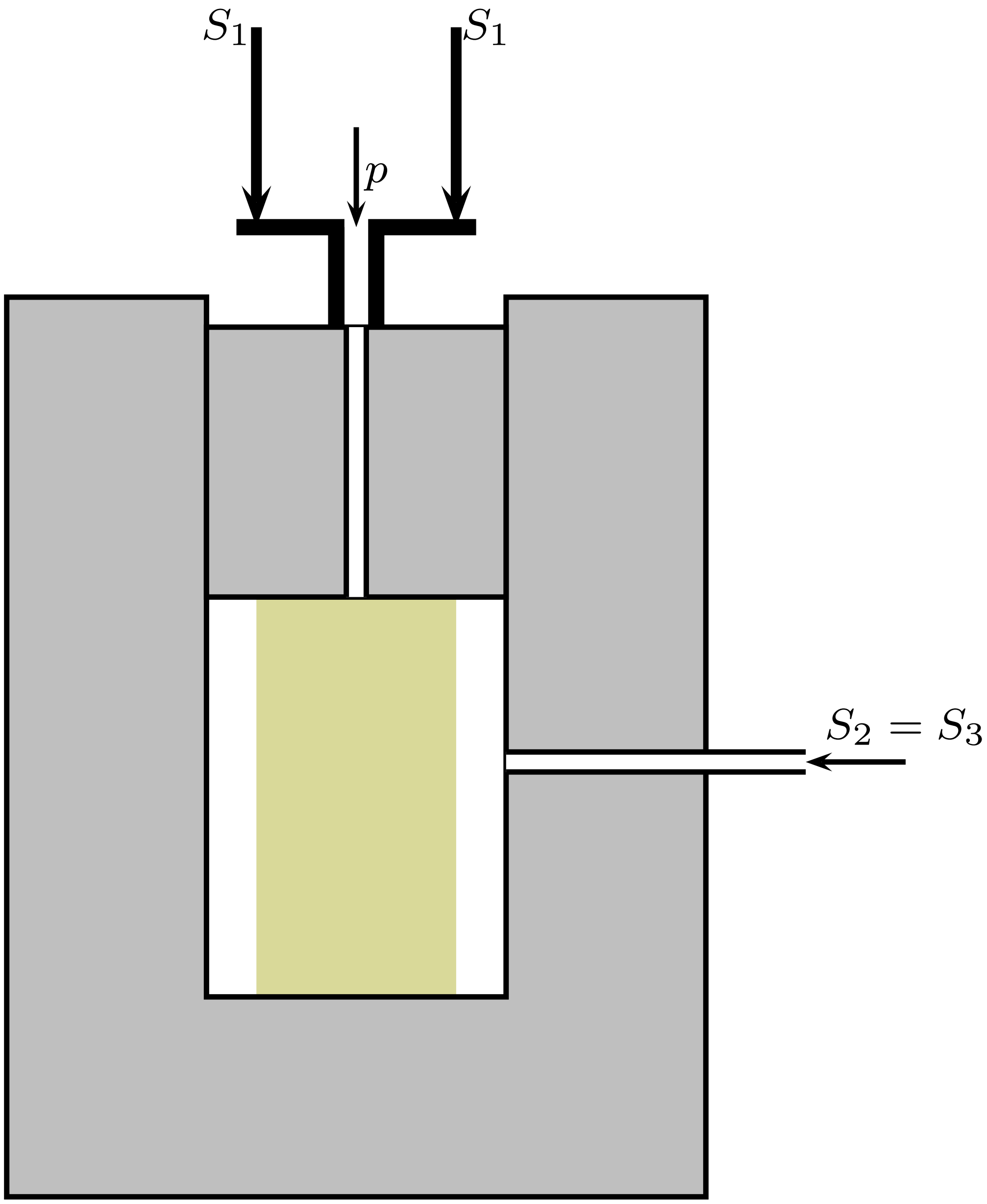
Compressibility Measurements

The unjacketed triaxial rock compressibility measurement. The confining pressure,

$$p_c = -\sigma = \frac{1}{3}(S_1 + S_2 + S_3),$$

is applied to all sides of the sample. The tube in the piston is used to regulate the pore pressure, p. Both the confining pressure and the fluid pressure are changed at the same time, so that their difference, $p_d = p_c - p$, remains constant.

$$\beta_s := -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{p_d,T} = \frac{1}{K_s}$$



Porous Rock Compressibilities

We can measure the following three compressibilities:

$$\beta := -\frac{1}{V}\left(\frac{\partial V}{\partial p_c}\right)_{p,T} = \frac{1}{K} \qquad \left(\text{Biot}: +\left.\frac{\delta\epsilon}{\delta\sigma}\right|_{\delta p=0} \equiv \frac{1}{K}\right)$$

$$\beta_s := -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{p_d,T} = \frac{1}{K_s} \qquad \left(\text{Biot}: +\left.\frac{\delta\epsilon}{\delta p}\right|_{\delta\sigma=0} \equiv \frac{1}{H}\right)$$

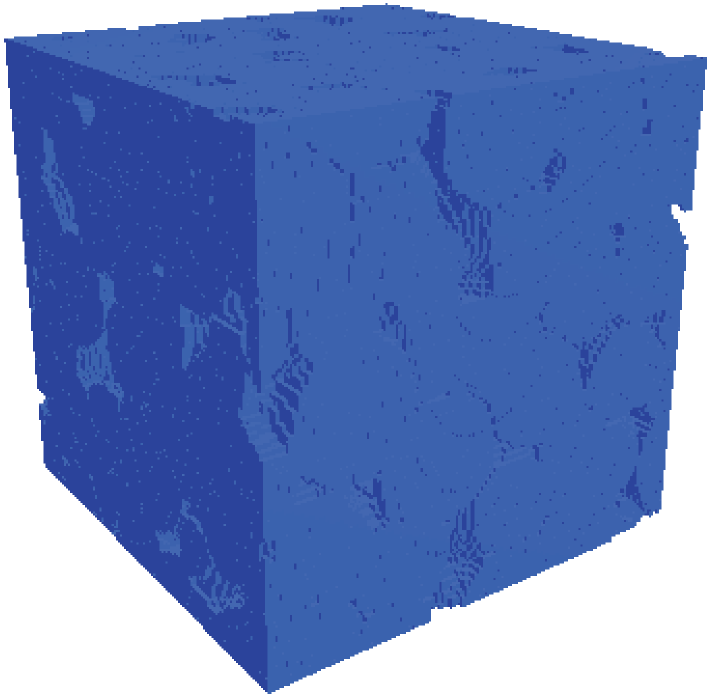
$$\beta_\phi := -\frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial p}\right)_{p_d,T} = \frac{1}{K_\phi} \qquad \left(\text{Biot}: +\left.\frac{\delta\zeta}{\delta p}\right|_{\delta\sigma=0} \equiv \frac{1}{R} = S_\sigma\right)$$

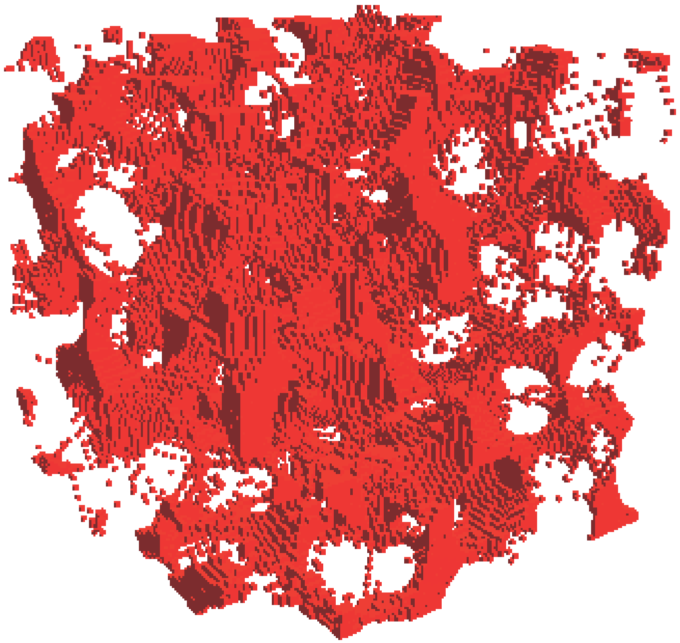
where V is the bulk volume of the sample, V_ϕ is the pore space volume
A fourth compressibility may be defined as

$$\beta_p := -\frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial p_c}\right)_{p,T} = \frac{1}{K_p} \qquad \left(\text{Biot}: +\left.\frac{\delta\zeta}{\delta\sigma}\right|_{\delta p=0} \equiv \frac{1}{H}\right)$$

but it depends on the porosity and the first two compressibilities above

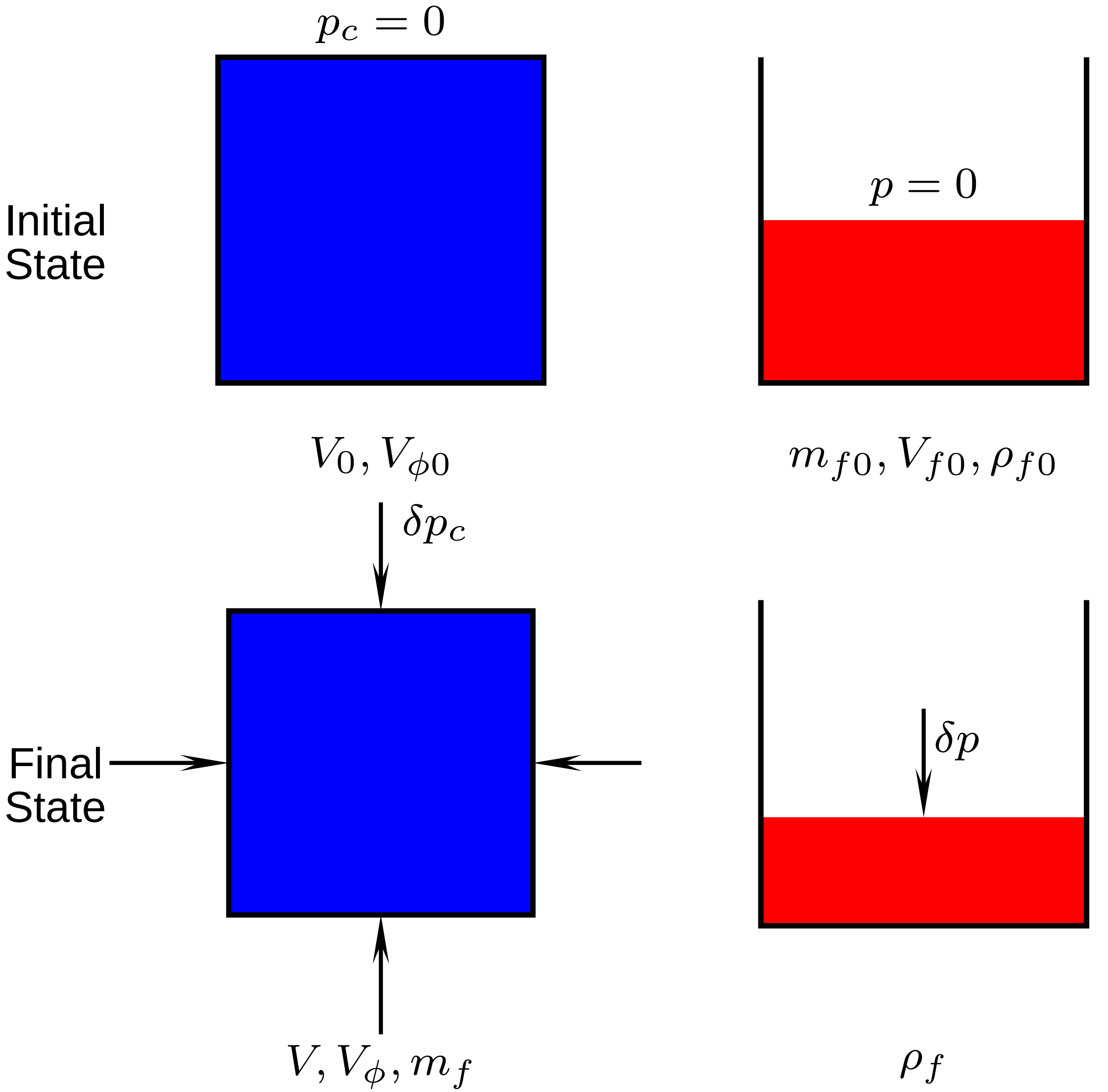
Porous Rock





At the reference state, we imagine a colored rock grain sample, in blue, filled with colored water, in red. First, we remove the red water into a beaker and fill the pore space with ordinary water. Second, we change the stress on the solid and the pore pressure, and “measure” the new pore volume, V_ϕ. Third, we measure the new red water volume under the new pore pressure, V_f. In general, the new pore volume and water volume will not be equal to each other, and water will have to flow in/out of the blue rock volume.

Biot's Increment of Fluid Mass ζ



Initially $V_{f0} = V_{\phi 0}$; the pore space is fully saturated with red fluid

Biot's Increment of Fluid Mass ζ

At the final state

$$m_f = m_{f0}\frac{V_\phi}{V_f}$$

After Biot, I will introduce the increment of fluid mass per unit initial bulk volume V_0, normalized by the initial fluid density m_{f0}/V_{f0}:

$$\zeta := \frac{\delta m_f/\rho_{f0}}{V_0} = \left(\frac{V_{f0}}{V_0}\right)\delta\left(\frac{V_\phi}{V_f}\right) = \frac{V_{f0}}{V_0}\ \frac{\delta V_\phi V_{f0} - \delta V_f V_{\phi 0}}{V_{f0}^2}$$

$$\zeta = \frac{1}{V_0}(\delta V_\phi - \delta V_f) = \phi_0(\epsilon_\phi - \epsilon_f)$$

Talk Outline. . .

- Refresher of Biot's static poroelasticity model
- Biot's dynamic poroelastic model from the non-equilibrium filtration theory
- Low frequency reflections from a plane interface between an elastic and an elastic fluid-saturated layers
- Different asymptotic regimes of the low-frequency reflections
- Conclusions

Biot Theory…

- The isotropic, permeable porous rock, and the pore-filling fluid are in mechanical equilibrium
- The stress is positive when it is tensile
- The fluid pressure is positive
- The state of rock and the fluid is described by the total stress on the bulk material, σ_{ij}, and the fluid pressure field p (σ_{ij} is the total force in direction i, acting on the surface element whose normal is in direction j)
- Following BIOT, in one spatial dimension, the small fluctuations of the total stress tensor, $\delta\sigma$, and of the fluid pressure, δp, will be called σ and p

Biot Theory...

$$\epsilon \equiv \frac{\delta V}{V_0} = \frac{1}{K}\sigma + \frac{1}{H}p$$ volumetric strain

$$\zeta \equiv \frac{\delta m_f}{V_0 \rho_{f_0}} = \frac{1}{H}\sigma + \frac{1}{R}p$$ fluid volume per unit volume

$$\left.\frac{\epsilon}{\sigma}\right|_{p=0} \equiv \frac{1}{K}$$ drained material compressibility

$$\left.\frac{\zeta}{\sigma}\right|_{p=0} = \left.\frac{\epsilon}{p}\right|_{\sigma=0} \equiv \frac{1}{H}$$ poroelastic expansion coefficient

$$\left.\frac{\zeta}{p}\right|_{\sigma=0} \equiv \frac{1}{R} = S_\sigma$$ unconstrained specific storage

Biot Theory…

$$-\left.\frac{p}{\sigma}\right|_{\zeta=0} \equiv B = \frac{R}{H}$$ SKEMPTON's coefficient

$$\left.\frac{\zeta}{p}\right|_{\epsilon=0} \equiv \frac{1}{M} = S_\epsilon$$ constrained specific storage

$$S_\epsilon = S_\sigma - \frac{K}{H^2}$$

$$\frac{K}{H} \equiv \alpha$$ BIOT-WILLIS' coefficient

$$\zeta = \alpha\epsilon + \frac{1}{M}p$$

Biot Theory...

- The poroelastic expansion coefficient $1/H$ has no analog in elasticity
- It describes how much a change of pore pressure also changes the bulk volume, while the applied stress is held constant
- $1/H$, and two other constants, K – drained bulk modulus, and the unconstrained storage coefficient S_σ, completely describe the linear, poroelastic response to volumetric deformation
- Other constants, such as SKEMPTON's coefficient, or BIOT-WILLIS' coefficient can be derived from the three fundamental BIOT constants

Definitions...

p	pressure increment, Pa
σ	stress increment, Pa
u	displacement of skeleton grains, m
u_t	velocity of displacement of skeleton grains, m/s
w	superficial displacement of fluid relative to solid, m
W	w_t Darcy velocity of fluid relative to solid, m/s
β	isothermal compressibility, Pa^{-1}
ϱ	$(1-\phi)\rho_g$, “dry” bulk density, kgm^{-3}
ϱ_b	$(1-\phi)\rho_g + \phi\rho_f$, bulk density, kgm^{-3}
ϵ	$\delta V/V$, increment of volumetric strain
ε	small parameter in series expansions
ζ	$\delta m_f/\rho_{f_0}/V_0$, increment of fluid content per unit volume

The Bulk Momentum Balance...

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \Big(\underbrace{\varrho_b \boldsymbol{u}_t}_{\text{solid+liquid momentum}} + \underbrace{\varrho_f \boldsymbol{W}}_{\text{relative liquid momentum}} \Big)\, dV$$

$$= \oint_{\delta V} \underbrace{\boldsymbol{\sigma}}_{\text{total stress}} \cdot \boldsymbol{n}\, dA + \int_V \underbrace{\boldsymbol{F}_b}_{\text{body force}}\, dV$$

- Small perturbation from equilibrium
- Incremental body force is zero

$$\frac{\partial}{\partial t}\big(\varrho_b \boldsymbol{u}_t + \varrho_f \boldsymbol{W}\big) = \nabla \cdot \boldsymbol{\sigma}$$

The Bulk Momentum Balance...

- Almost incompressible grains ($\alpha \approx 1$)
- Poroelastic effective stress $\boldsymbol{\sigma}'$, and Terzaghi effective stress are equal
- 1D normal deformations, $\boldsymbol{\sigma} = \sigma_{xx}$

$$\frac{\partial}{\partial t}\big(\varrho_b u_t + \varrho_f W\big) = \frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial \sigma'_{xx}}{\partial x} - \frac{\partial p}{\partial x}$$

$$\sigma'_{xx} \approx K \frac{\partial u}{\partial x} = \frac{1}{\beta}\frac{\partial u}{\partial x}$$

- K is the drained bulk modulus

Force Balance...

- The second Newton's law for the bulk solid is

$$\varrho_b \partial_{tt} u + \varrho_f \partial_t W = \frac{1}{\beta} \partial_{xx} u - \partial_x p \qquad (1)$$

Darcy's Law...

- Consider steady state, single-phase flow of an almost incompressible fluid
- The superficial fluid velocity relative to the solid

$$W = -\frac{\kappa}{\eta}\frac{\partial \Phi}{\partial x}$$

- In horizontal flow, viewed from a non-inertial coordinate system moving with the solid, the differential of the flow potential is

$$\underbrace{d\Phi}_{\text{Mechanical energy}} = \underbrace{dp}_{\text{Viscous dissipation}} + \underbrace{\varrho_f \partial_{tt} u \, dx}_{\text{Inertial force}}$$

Extended Darcy's Law…

- In time-dependent, single-phase flow, we can write

$$\frac{\partial W}{\partial t} \approx \frac{W_{\text{future}} - W}{\tau}$$

where W_{future} is a future value of Darcy's velocity, and τ is a characteristic time of transition

- At constant position $\boldsymbol{x}$, and constant value of W_{future}, we can integrate

$$W_{\text{future}} - W \propto \exp\left(\frac{-t}{\tau}\right)$$

- Therefore, τ is a characteristic relaxation time for transient flow, *e.g.*, JAMES C. MAXWELL, 1867

Extended Darcy's Law. . .

- In time-dependent, single-phase flow, we now write

$$\boldsymbol{W}_{\text{future}} \approx \boldsymbol{W} + \frac{\partial \boldsymbol{W}}{\partial t}\tau + \cdots = -\frac{\kappa}{\eta}\nabla\Phi$$

- This is the essence of ALISHAEV's, and BARENBLATT & VINNICHENKO's extension of DARCY's law
- Dimensional analysis suggests that

$$\tau = \eta\beta_f F(\kappa/L^2)$$

where L is the characteristic length scale of REV

Extended Darcy's Law. . .

We characterize the dynamics of horizontal fluid flow in a non-inertial coordinate system as follows

$$W + \tau \frac{\partial W}{\partial t} = -\frac{\kappa}{\eta}\frac{\partial p}{\partial x} - \varrho_f \frac{\kappa}{\eta}\frac{\partial^2 u}{\partial t^2} \quad (2)$$

Mass Balances & Isothermal EOS's. . .

- Slightly compressible fluid

$$\frac{\partial(\varrho_f \phi)}{\partial t} = -\frac{\partial}{\partial x}\left(\varrho_f W + \phi \varrho_f \frac{\partial u}{\partial t}\right)$$

$$\frac{d\varrho_f}{\varrho_f} = \beta_f dp$$

- Almost incompressible solid grains

$$\frac{\partial[\varrho_g(1-\phi)]}{\partial t} = -\frac{\partial}{\partial x}\left(\varrho_g(1-\phi)\frac{\partial u}{\partial t}\right)$$

$$\frac{1}{\varrho_g} d\varrho_g = \beta_{gs} d\sigma_x + \beta_{gf} dp$$

$\beta_{gs} \ll \beta$ and $\beta_{gf} \ll \beta_f$

Reduced Mass Balances...

- With almost incompressible grains, the bulk deformation occurs only through the porosity change
- With some algebra, the mass balance equations reduce to

$$\frac{\partial^2 u}{\partial x \partial t} + \phi \beta_f \frac{\partial p}{\partial t} = -\frac{\partial W}{\partial x} \qquad (3)$$

- Note that we now have three unknowns u, p and W, and three balance equations: (1) Force balance of bulk solid, (2) Force balance in viscous-dominated fluid flow, and (3) Combined mass balance of fluid and solid

The Governing Equations...

For a linearly compressible rock skeleton and fluid, and small perturbations from thermodynamic equilibrium:

Force balance of bulk material

$$\varrho_b \partial_{tt} u + \varrho_f \partial_t W = -\frac{1}{\beta} \partial_{xx} u - \partial_x p \qquad (1)$$

Force balance of viscous fluid

$$W + \tau \partial_t W = -\frac{\kappa}{\eta} \Big(\partial_x p - \varrho_f \partial_{tt} u \Big) \qquad (2)$$

F/S mass balances + EOS's

$$\phi \beta_f \partial_t p = -\partial_x \left(W + \partial_t u \right) \qquad (3)$$

Biot's Theory. . .

We define the superficial fluid displacement

$$W := \partial_t w \tag{4}$$

and insert it into mass balance equation (3)

$$\phi\beta_f \partial_t p = -\partial_{xt}(w + u)$$

By integration in t and differentiation in x, we obtain

$$\partial_x p = -\frac{1}{\phi\beta_f}\partial_{xx}(u + w) \tag{5}$$

Now we substitute the displacement (4) and the final result (5) into the governing equations

Biot's Theory...

- Our equations

$$\varrho_b \frac{\partial^2 u}{\partial t^2} + \varrho_f \frac{\partial^2 w}{\partial t^2} = \left(\frac{1}{\beta} + \frac{1}{\phi\beta_f}\right)\frac{\partial^2 u}{\partial x^2} + \frac{1}{\phi\beta_f}\frac{\partial^2 w}{\partial x^2}$$

$$\varrho_f \frac{\partial^2 u}{\partial t^2} + \tau\frac{\eta}{\kappa}\frac{\partial^2 w}{\partial t^2} = \frac{1}{\phi\beta_f}\frac{\partial^2 u}{\partial x^2} + \frac{1}{\phi\beta_f}\frac{\partial^2 w}{\partial x^2} - \frac{\eta}{\kappa}\frac{\partial w}{\partial t}$$

- Biot's 1962 equations

$$\frac{\partial^2}{\partial t^2}\left(\varrho_b u + \varrho_f w\right) = \frac{\partial}{\partial x}\left(A_{11}\frac{\partial u}{\partial x} + M_{11}\frac{\partial w}{\partial x}\right)$$

$$\frac{\partial^2}{\partial t^2}\left(\varrho_f u + m w\right) = \frac{\partial}{\partial x}\left(M_{11}\frac{\partial u}{\partial x} + M\frac{\partial w}{\partial x}\right) - \frac{\eta}{\kappa}\frac{\partial w}{\partial t}$$

Biot's Theory...

- We have assumed an isotropic porous medium and incompressible grains

The Biot-Willis coefficient $\alpha = K/H \approx 1$

The undrained bulk modulus $K_u = K + K_f/\phi$

- The Biot coefficients are then constant and equal to

$$A_{11} = K_u \approx \frac{1}{\beta} + \frac{1}{\phi\beta_f} \quad \text{and} \quad M_{11} = M = K_u B \approx \frac{1}{\phi\beta_f}$$

where $B = R/H$ is Skempton's coefficient, $1/H$ being the poroelastic expansion coefficient, and $1/R$ the unconstrained specific storage coefficient

Biot's Theory…

- The dynamic coupling coefficient in Biot's theory, m, is equal to the inverse fluid mobility, η/κ
- The dynamic coupling coefficient is often expressed through the tortuosity factor T: $m = T\varrho_f/\phi$
- Hence, for the tortuosity and relaxation time, we obtain the following relationship:

$$T = \tau \underbrace{\frac{\eta\phi}{\kappa\varrho_f}}_{\text{Inv. kinematic mobility}} \quad \text{or} \quad \tau = T\frac{\kappa\varrho_f}{\eta\phi} \tag{6}$$